# Lecture 10. Invertible matrices

**Def** (1) The <u>$n\times n$ identity matrix</u> is the matrix $I_n$ whose columns are the standard basis vectors $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n$.

e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2) The <u>inverse</u> of an $n\times n$ matrix $A$ is the matrix $A^{-1}$ with $AA^{-1} = A^{-1}A = I_n$.

**Note** (1) The inverse of an $n\times n$ matrix is an $n\times n$ matrix.

(2) The inverse of a nonsquare matrix is undefined.

(3) We have $I_nA = AI_n = A$ for any $n\times n$ matrix $A$.

(cf. $1\cdot a = a\cdot 1 = a$ for any $a \in \mathbb{R}$)

(4) We often write $I$ in place of $I_n$.

**Def** (1) The <u>identity transformation</u> on $\mathbb{R}^n$ is the linear transformation $\mathbb{1}_n : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $\mathbb{1}_n(\vec{x}) = \vec{x}$.

(2) The <u>inverse</u> of a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the linear transformation $T^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $T \circ T^{-1} = T^{-1} \circ T = \mathbb{1}_n$.

**Note** (1) The standard matrix of $\mathbb{1}_n$ is $I_n$.

(2) A linear transformation $T$ has an inverse.

$\iff$ $T$ is <u>injective and surjective</u>

"bijective"

**Prop** If a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with standard matrix $A$ has an inverse $T^{-1}$, the standard matrix of $T^{-1}$ is $A^{-1}$.

$T \circ T^{-1} = T^{-1} \circ T = \mathbb{1}_n \iff AA^{-1} = A^{-1}A = I_n$.

**Thm** A square matrix $A$ has an inverse $\iff$ $RREF(A) = I$

**pf** Take the linear transformation $T$ with standard matrix $A$

$A$ has an inverse

$\iff$ $T$ has an inverse

$\iff$ $T$ is injective and surjective

$\iff$ $RREF(A)$ has a leading 1 in every row and column

$\iff$ $RREF(A) = I$

**Prop** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary $2 \times 2$ matrix.

(1) If we have $ad - bc = 0$, then $A^{-1}$ does not exist.

(2) Otherwise, we have $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

**Prop** Let $A$ be a square matrix with an inverse $A^{-1}$. If $B$ is the matrix formed by concatenating $A$ and $I$, then $RREF(B)$ is formed by concatenating $I$ and $A^{-1}$.

$B = [\, A \mid I \,] \Rightarrow RREF(B) = [\, I \mid A^{-1} \,]$

**Ex** For each linear transformation, find the standard matrix of its inverse if it exists.

(1) $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which maps $\vec{e}_1$ to $\vec{e}_1 - 3\vec{e}_2$ and $\vec{e}_2$ to $\vec{e}_2$.

**Sol** $T_1(\vec{e}_1) = \vec{e}_1 - 3\vec{e}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $T_1(\vec{e}_2) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ The standard matrix of $T_1$ is

$$A_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

which has an inverse $(1 \cdot 1 - (-3) \cdot 0 = 1 \neq 0)$

$\Rightarrow$ $T_1$ has an inverse with standard matrix

$$A_1^{-1} = \frac{1}{1 \cdot 1 - (-3) \cdot 0} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}$$

(2) $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

$$T_2(\vec{x}) = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix} \quad \text{for } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**Sol** The standard matrix of $T_2$ is

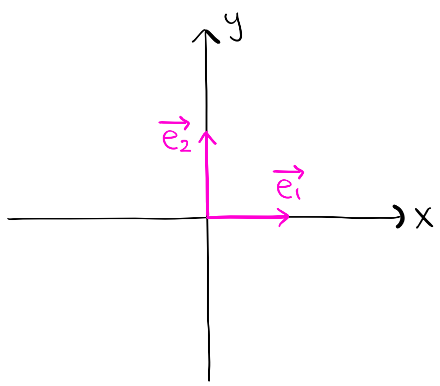
$$A_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

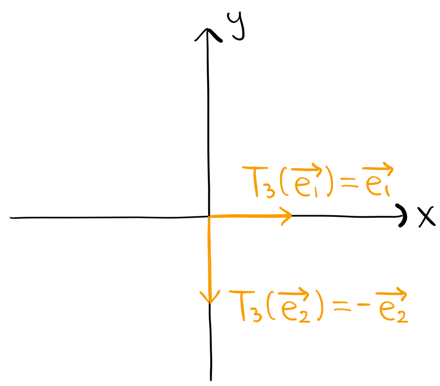
which does not have an inverse $(1 \cdot 1 - (-1) \cdot (-1) = 0)$

$\Rightarrow$ $T_2$ does not have an inverse

(3) $T_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which reflects each vector through the x-axis.

Sol




$$T_3(\vec{e}_1) = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad T_3(\vec{e}_2) = -\vec{e}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$\Rightarrow$ The standard matrix of $T_3$ is

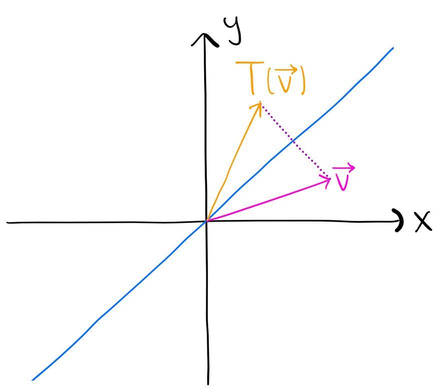
$$A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which has an inverse $(1\cdot(-1) - 0\cdot 0 = -1 \neq 0)$

$\Rightarrow$ $T_3$ has an inverse with standard matrix

$$A_3^{-1} = \frac{1}{1\cdot(-1)-0\cdot 0}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \; (= A_3)$$

Note In fact, if $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a reflection through a line, it is invertible with $T^{-1} = T$.




$T(T(\vec{v})) = \vec{v}$ for any $\vec{v} \in \mathbb{R}^2$

$\Rightarrow T \circ T = \mathbb{1}$

$\Rightarrow T^{-1} = T.$

(4) $T_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which rotates each vector about the origin through $\frac{\pi}{4}$ radians

Sol The standard matrix of $T_4$ is

$$A_4 = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

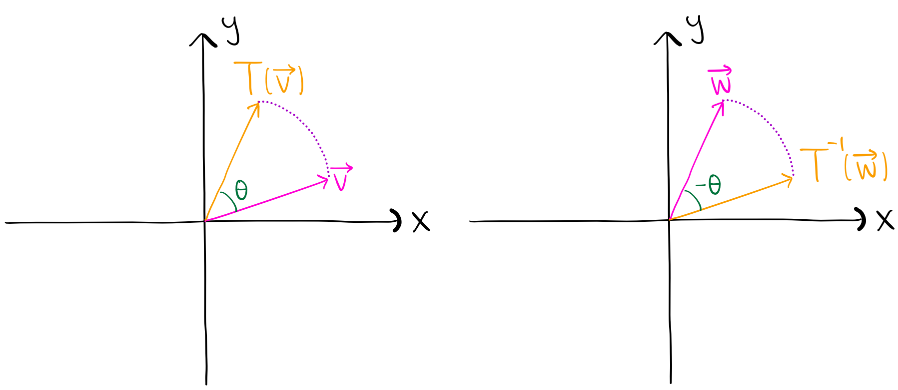
which has an inverse $\left( 1/\sqrt{2} \cdot 1/\sqrt{2} - (-1/\sqrt{2}) \cdot 1/\sqrt{2} = 1 \neq 0 \right)$

$\Rightarrow$ $T_4$ has an inverse with standard matrix

$$A_4^{-1} = \frac{1}{1/\sqrt{2} \cdot 1/\sqrt{2} - (-1/\sqrt{2}) \cdot 1/\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \boxed{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

Note In fact, if $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a rotation about the origin through $\theta$ radians, it is invertible with $T^{-1}$ being the rotation about the origin through $-\theta$ radians.

$\theta$ rads clockwise




$T \circ T^{-1} = T^{-1} \circ T = \mathbb{1}$ (rotation through $\theta - \theta = 0$ radians)

(5) $T_5 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with standard matrix

$$A_5 = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 2 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

Sol Take the matrix formed by concatenating $A_5$ and $I$.

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -4 & 9 & -6 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 3 & -6 & 4 \end{array}\right]$$

$A_5$ $\quad$ $I$ $\qquad$ $\text{RREF}(A_5) = I$ $\quad$ $A_5^{-1}$

$A_5$ has an inverse $(\text{RREF}(A_5) = I)$

$\Rightarrow$ $T_5$ has an inverse with standard matrix

$$A_5^{-1} = \boxed{\begin{bmatrix} -4 & 9 & -6 \\ -1 & 2 & -1 \\ 3 & -6 & 4 \end{bmatrix}}$$

(6) $T_6 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with standard matrix

$$A_6 = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}$$

Sol Take the matrix formed by concatenating $A_6$ and $I$.

$$\left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 2 & -3 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

$A_6$ $\quad$ $I$ $\qquad$ $\text{RREF}(A_6)$

$A_6$ does not have an inverse $(\text{RREF}(A_6) \neq I)$

$\Rightarrow$ $T_6$ $\boxed{\text{does not have an inverse}}$